13/10/23

MATH2050A Tutorial Planning

<u>Topic</u>: Monotone Convergence Thm:

<u>Recall</u>: $\{x_n\}$ is increasing if $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$.

decreasing if $x_n \geq x_{n-1}$ for all $n \in \mathbb{N}$.

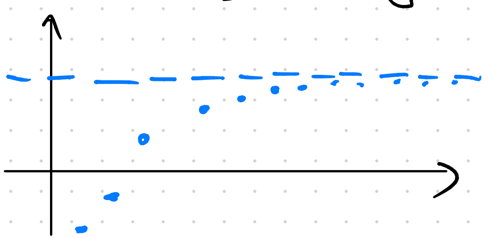
monotone if it is increasing or decreasing.

<u>MCT</u>: A monotone sequence $\{x_n\}$ is convergent iff it is bounded.

Case 1: $\{x_n\}$ increasing: $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

Case 2: $\{x_n\}$ decreasing: $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.

Picture:



<u>Pf</u>: $\Rightarrow$: WTS $\{x_n\}$ is bounded. Let $x = \lim_{n\to\infty} x_n$. Then for each n,

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x|.$$

$\exists N$ s.t. $\forall n \geq N$, $|x_n - x| < 1$. So for $n \geq N$, $|x_n| \leq 1 + |x|$.

Now let $M := \sup\{|x_1|, \ldots, |x_{N-1}|, 1+|x|\}$.

Then $|x_n| \leq M$ for all M and $\{x_n\}$ is bounded.

$\Leftarrow$: WTS $\{x_n\}$ converges.

Case 1: $\{x_n\}$ is bounded, increasing. By boundedness, $\exists M \in \mathbb{R}$ s.t. $x_n \leq M$ $\forall n \in \mathbb{N}$.

Since $\{x_n\}$ is bounded, non-empty, $x^* := \sup\{x_n\}$ exists.

WTS $x^* = \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$ be given. $x^* - \varepsilon$ is not an u.b. of $\{x_n\}$, so

$\exists K$ s.t. $x^* - \varepsilon < x_K \leq x_n$ for all $n \geq K$.

So we have shown that given $\varepsilon > 0$, $\exists K$ s.t. $\forall n \geq K$,

$$x^* - \varepsilon < x_K \leq x_n \underset{\text{sup}}{\leq} x^* < x^* + \varepsilon$$

$\Rightarrow$ $|x_n - x| < \varepsilon$. for all $n \geq K$.

Case 2: $\{x_n\}$ is bounded, decreasing. Apply case (1) to $\{-x_n\}$.

Applications:

1). Harmonic series $h_n := \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$. ← is $\{h_n\}$ convergent or divergent?

$h_n = 11.4$ for $n = 50{,}000$

$h_n = 12.1$ for $n = 100{,}000$

$h_n > 50$ for $n \approx 5.2 \times 10^{21}$.

Clearly h_n is increasing. So by MCT, suffices to check whether h_n is bounded

$$h_{2^n} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \dots + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}\right).$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \dots + \underbrace{\left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right)}_{2^{n-1} \text{ of these.}}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$$

$= 1 + \frac{n}{2}$ which is unbounded by AP.

So $\{h_n\}$ diverges, just extremely slowly!

2) Defining e:

$$e_n := \left(1+\frac{1}{n}\right)^n \overset{\text{binomial thm}}{=} 1 + \frac{\cancel{n}}{1!}\cdot\frac{1}{\cancel{n}} + \frac{\cancel{n}(n-1)}{2!}\cdot\frac{1}{n^{\cancel{2}}} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(2)\cdot 1}{n!}\,\frac{1}{n^n}$$

$$= \underbrace{1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}_{n+1 \text{ terms}}.$$

$$e_{n+1} = \underbrace{1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}_{n+2 \text{ terms}}.$$

each term in e_n is less than or equal to the corresponding term in e_{n+1}

So $\{e_n\}$ is increasing. Clearly $2 < e_n$ for all $n \geq 1$.

For $p = 1, 2, \ldots, n$, $1 - \frac{p}{n} < 1$, $2^{p-1} \leq p! \Rightarrow \frac{1}{p!} \leq \frac{1}{2^{p-1}}$.

So $e_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3$.

So by MCT, $\{e_n\}$ converges to $e := \sup\{e_n\}$. ← define e w/out knowing its value explicitly ✓.

3) Recurrence Relations:

a) $x_1 = \frac{3}{2}$, $x_{n+1} = 2 - \frac{1}{x_n}$.

By induction: $1 < x_n \leq 2$ for $n \geq 2$: $x_2 = 2 - \frac{1}{x_n} > 1$ and $x_2 \leq 2$. base case ✓.

Sps $1 < x_k \leq 2$ for some k. Then

$$x_{k+1} = 2 - \frac{1}{x_k} > 2 - 1 = 1., \quad x_{k+1} = 2 - \frac{1}{x_k} \leq 2 - \frac{1}{2} \leq 2.$$

$$x_k > 1 \Rightarrow \frac{1}{x_k} < 1 \Rightarrow -\frac{1}{x_k} > -1$$

$$x_k \leq 2 \Rightarrow \frac{1}{x_k} \geq \frac{1}{2} \Rightarrow -\frac{1}{x_k} \leq -\frac{1}{2}.$$

So $\{x_n\}$ is bounded.

$\{x_n\}$ is decreasing: Sps $x_k \geq x_{k+1}$ for some k, then $\frac{1}{x_k} \leq \frac{1}{x_{k+1}}$

$$x_{k+1} = 2 - \frac{1}{x_k} \geq 2 - \frac{1}{x_{k+1}} = x_{k+2}.$$

$$\Rightarrow -\frac{1}{x_k} \geq -\frac{1}{x_{k+1}}$$

Base Case: $x_1 = \frac{3}{2}$, $x_2 = 2 - \frac{2}{3} = \frac{6-2}{3} = \frac{4}{3} < \frac{3}{2}$.

So by MCT $\{x_n\}$ converges.

Moreover, $x = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{n+1}$. So x satisfies

$$x = \lim_{n\to\infty} x_{n+1} = 2 - \frac{1}{\lim\limits_{n\to\infty} x_n} = 2 - \frac{1}{x}.$$

$\Rightarrow$ x satisfies $x^2 - 2x + 1 = 0$. $\Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$ //.

b) Try $x_{n+1} := \sqrt{2x_n}$, $x_1 = 1$.

4) Show $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ converges using MCT.

(Hint: if $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$).

Pf: Clearly $\{x_n\}$ is increasing.

Boundedness: $x_1 \leq 2$., for $n \geq 2$,

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1^2} + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right).$$

$$= 2 - \frac{1}{n} \leq 2.$$

So $x_n \leq 2$ for all n.

Hence by MCT, $\{x_n\}$ converges. //